
# Algorithms for Data Science
# Finding Similar Items

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M2 Data Science

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# Similar Items










# General Problem

Many data mining tasks can be expressed as finding **similar** sets.

- same as **finding near-neighbours in high-dimensional space**

Some applications:

- **similar pages on the Web**: duplicate detection for search engines
- **customer who purchased similar products**
- **images having similar features**

# Similarity and Distance

**Input** – set of high dimensional data points represented as vectors $(x_1\ x_2\ x_3\ \ldots\ x_n)$ and a distance function $d(p_i, p_j)$

**Problem** – find pairs of data points $(p_i, p_j)$ that are close, i.e., in a distance threshold $d(p_i, p_j) \leqslant \tau$

- comparing all pairs would take $\mathcal{O}(N^2)$ ($N$ number of data points) – too expensive
- can be done much faster, around $\mathcal{O}(N)$

# Documents and Set Similarity

In this lecture, we will study how to find **similar documents** – **near-duplicate pairs**

- plagiarism, mirror pages, articles having the same source

Documents are represented as **sets** / **bags** – we will discuss *how*

Focus on **Jaccard** distance/similarity:

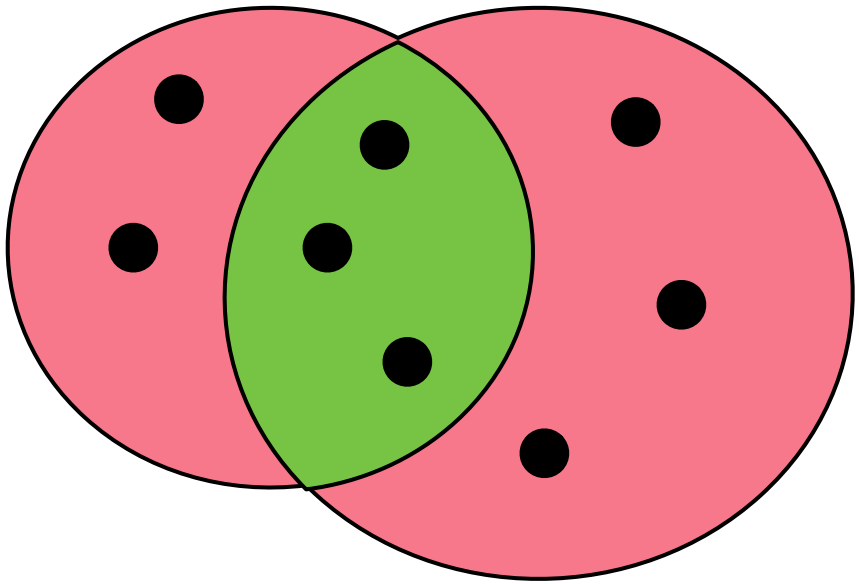
- **Jaccard similarity** of two sets $S_1$, $S_2$:

$$\text{sim}(S_1, S_2) = \frac{|S_1 \cap S_2|}{|S_1 \cup S_2|}$$

- **Jaccard distance** is:

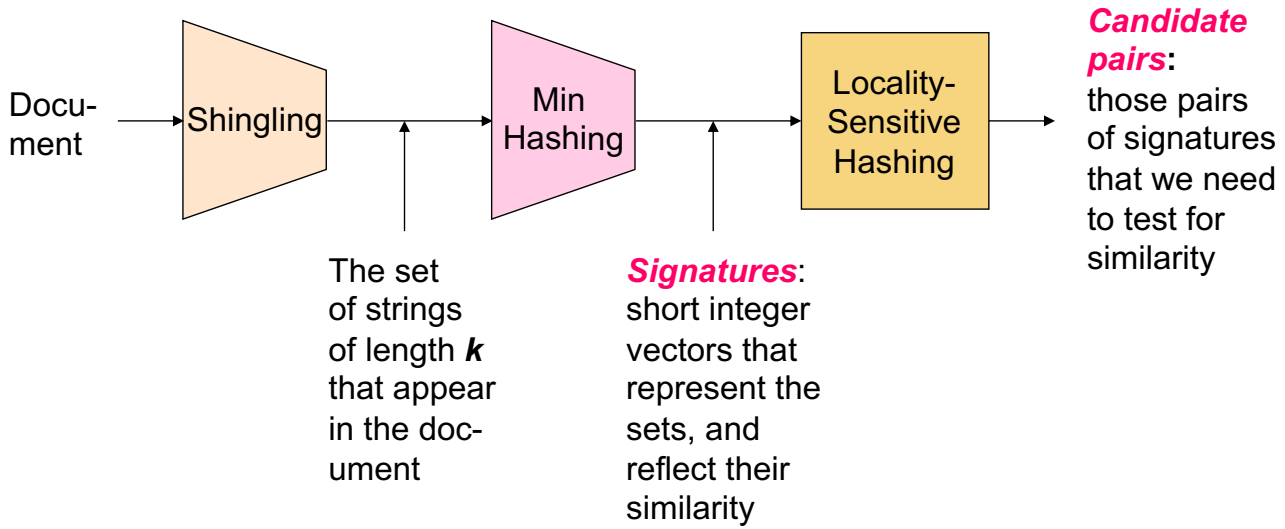
$$d(S_1, S_2) = 1 - \text{sim}(S_1, S2)$$

# Jaccard similarity/distance



1. similarity $3/8$ – *fraction* of the green area
2. distance $5/8$ – *fraction* of the red area

# Steps for Finding Similar Documents




1. **shingling** – converting documents to sets
2. **min-hashing** – convert each document to a short signature
3. **locality-sensitive hashing** – reduce the number of pairs of signatures to compare

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# Shingling

*Naïve way* – represent documents as the set of their words – would find many documents that are similar (common words in the language of the document)

- better way – shingling

**Shingling**: $k$-shingle = any substring of length $k$ found in the document

- the document is then the set of shingles appearing at least once

*Example*

- take the document $D$ represented by the string `abcdabd`
- the set of 2-shingles is then $\{\texttt{ab}, \texttt{bc}, \texttt{cd}, \texttt{da}, \texttt{bd}\}$

# Shingling in Practice

**Principle** – $k$ should be picked large enough that the probability of any given shingle appearing in any given document is as low as possible

- assume a document has the $27$ chars in the ASCII character set and $k = 5$
- the number of shingles is $27^5 = 14{,}348{,}907$ possible shingles – so $k = 5$ works well for any document that is much smaller than the above size

# Shingling in Practice

- in practice, $k = 5$ is good for emails, $k = 10$ is good for large documents
- the size of the sets can be larger that the documents – hash the shingles to an integer having a limited number of bits – e.g., for $k = 2$ we only need 10 bits:

$$\{\texttt{ab}, \texttt{bc}, \texttt{cd}, \texttt{da}, \texttt{bd}\} \rightarrow \{342, 825, 312, 54\}$$

- the similarity/distance is the Jaccard similarity of sets, applied on the $k$-shingle sets of each document

# Representing Sets of Documents as a Matrix

Conceptually, we will represent the sets of documents as a **Boolean matrix**

- **rows** are the indexes of the possible shingles
- **colums** represent the documents as

## Running example

Documents represented as sets $D_1 = \{a, d\}$, $D_2 = \{c\}$, $D_3 = \{b, d, e\}$, $D_4 = \{a, c, d\}$

Table:

| Shingle hash | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 1 | 0 | 1 |
| 4 | 1 | 0 | 1 | 1 |
| 5 | 0 | 0 | 1 | 0 |

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# Why Min-Hashing?

The encoding of sets as boolean values can still be **too costly** – cost = the number of different possible shingles, or the **size of the universal set**

We want to minimize the size of this set, and transform the set into a **signature set**

- in other words, compress the size of the **columns** (=documents) in the matrix

**Principle** – similarity of signature sets = similarity of shingle sets = similarity of documents

# Min-Hashing for Jaccard similarity

**Objective** – find a hash function $h$ (on the shingle set of the documents), such that:

- if $\text{sim}(D_1, D_2)$ is high, then with high probability, $h(D_1) = h(D_2)$
- if $\text{sim}(D_1, D_2)$ is low, then with high probability, $h(D_1) \neq h(D_2)$

Not all similarity metrics / distances have such a hash function!

- Jaccard has one – Min-Hashing

# Min-Hashing for Jaccard similarity

Hash each column $C$ of the table to a small signature $h(C)$:

1. $h(C)$ is **small enough to fit in main memory**
2. $\text{sim}(C_i, C_j)$ is **the same** as $\text{sim}(h(C_i), h(C_j))$

# Min-Hashing

Shuffle the rows of the matrix using a **random permutation** $\pi$

Define the hash function $h_\pi(C)$ as the **first row** (in permutation order of $\pi$) where we find a value of **1**

**Example**

Permutation:

| $\pi$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| 2 | 0 | 0 | **1** | 0 |
| 5 | 0 | 0 | 1 | 0 |
| 1 | **1** | 0 | 0 | **1** |
| 4 | 1 | 0 | 1 | 1 |
| 3 | 0 | **1** | 0 | 1 |

Min-Hash:

| $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|
| 1 | 3 | 2 | 1 |

# Min-Hash Property

| $D_1$ | $D_4$ |
|---|---|
| 1 | 1 |
| 0 | 0 |
| 0 | 1 |
| 1 | 1 |
| 0 | 0 |

$\text{sim}(D_1, D_4) = 2/3$

**Property** $\Pr[h_\pi(D_1) = h_\pi(D_2)] = \text{sim}(D_1, D_4)$, for any random permutation $\pi$

Proof sketch:

- let $s \in D$ a shingle
- equally likely that $s \in D$ is mapped to the min element; $\Pr[\pi(s) = \min(\pi(D))] = 1/|D|$
- let $s$ be such that $\pi(s) = \min(\pi(D_1 \cup D_4))$
- either $\pi(s) = \min(\pi(D_1))$ if $s \in D_1$, or $\pi(s) = \min(\pi(D_1))$ if $s \in D_4$
- probability that both are true is $\Pr[s \in D_1 \cap D_4]$
- $\Pr[h_\pi(D_1) = h_\pi(D_2)] = \frac{|D_1 \cap D_4|}{|D_1 \cup D_4|} = \text{sim}(D_1, D_4)$.

# Min-Hash in Practice

In practice, we need **multiple hash functions**, and thus the similarity of two documents is the fraction of the hash functions in which they agree

- this works because of **the min-hash property**, the similarity of columns is the same as the **expected** similarity of their signatures

## Implementation

- permuting rows is **too costly**!
- we can use well-chosen hash functions that achieve a permutation
- the more hash functions we choose, the more exact the computation is – but **more costly**

Signature of a document: $\mathcal{O}(K)$ (number of hash functions)

# Min-Hash in Practice

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $x+1 \mod 5$ | $3x+1 \mod 5$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 0 | 2 | 4 |
| 3 | 0 | 1 | 0 | 1 | 3 | 2 |
| 4 | 1 | 0 | 1 | 1 | 4 | 0 |
| 5 | 0 | 0 | 1 | 0 | 0 | 3 |

**Algorithm**: Choose $K$ permutation functions, initialize $\text{sig}(i, c) = \infty$, and then for each row(=shingle) $r$:

1. compute $h_1(r), \ldots, h_k(r)$
2. for each column(=document) $c$: if $c$ has 1, then set $\text{sig}(i, c) = \min(h_i(r), \text{sig}(i, c))$ for $i \in 1, \ldots, K$

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# Objectives

We achieved smaller documents, but we still need to find a way to compare as few pairs as possible

**Idea** find a way to only compare pairs that have a similarity above a threshold $t$

- LSH: use a function $f(x, y)$ that tell whether the pair $x, y$ is a candidate pair for comparison

# LSH for Min-Hashing

Assume we have a similarity threshold $s$

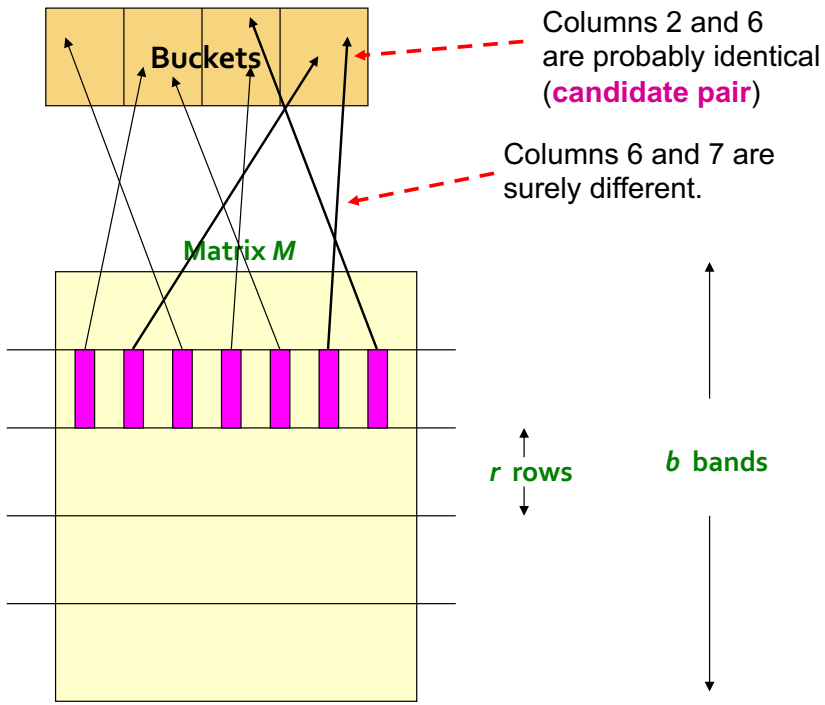
Columns $x$ and $y$ of the signature matrix $M$ are candidate pairs if the signature agrees on at least a fraction $s$ of their rows

- **reminder** we assume that the min-hashed signature output the same expected similarity as the real one

**Idea behing LSH for Min-Hashing**:

- hash columns of the signature matrix several times, so that only similar columns are likely to hash to the same bucket – candidate pairs are those that hash to the same bucket
- we can divide $M$ into $b$ bands of $r$ rows each

# LSH for Min-Hashing




- for each band, hash the portion of the column into $k$ buckets
- candidates are column pairs (=document pairs) hashing to the same bucket at least once
- have to tune $b$ and $r$ to catch most similar pairs, but fewer non-similar pairs

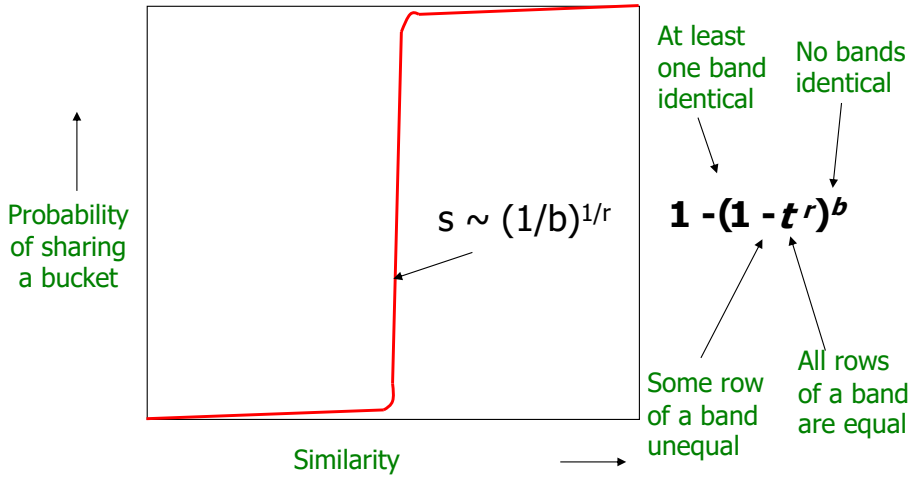
# Tuning $b$ and $r$

**Tradeoff** number of min-hashes, number of bands $b$, number of rows per band $r$

How to compute this?

1. prob. that signatures agree in all rows of one band is $s^r$
2. prob. that signatures disagree in at least one row is $1 - s^r$
3. the prob. that signatures disagree in at least one row of each of the bands is $(1 - s^r)^b$
4. **candidate pair** if agrees in all the rows of at least one band, prob. is $1 - (1 - s^r)^b$

# S-curve for LSH




have to choose the threshold roughly where the probability is $1/2$ – where the curve is steepest

**Approximate threshold** $t = (1/b)^{1/r}$

# Example

Say $D_1$ and $D_2$ are 80% similar, $b = 20$, $r = 5$

- prob. $D_1$,$D_2$ identical in a given band $0.8^5 = 0.328$
- prob. are not similar in any of the bands: $(1 - 0.328)^{20} = 0.00035$

Only 0.035% of the documents are false negatives (similar but they do not hash in the same bucket anywhere), 99.965% of true positives are found

# Example

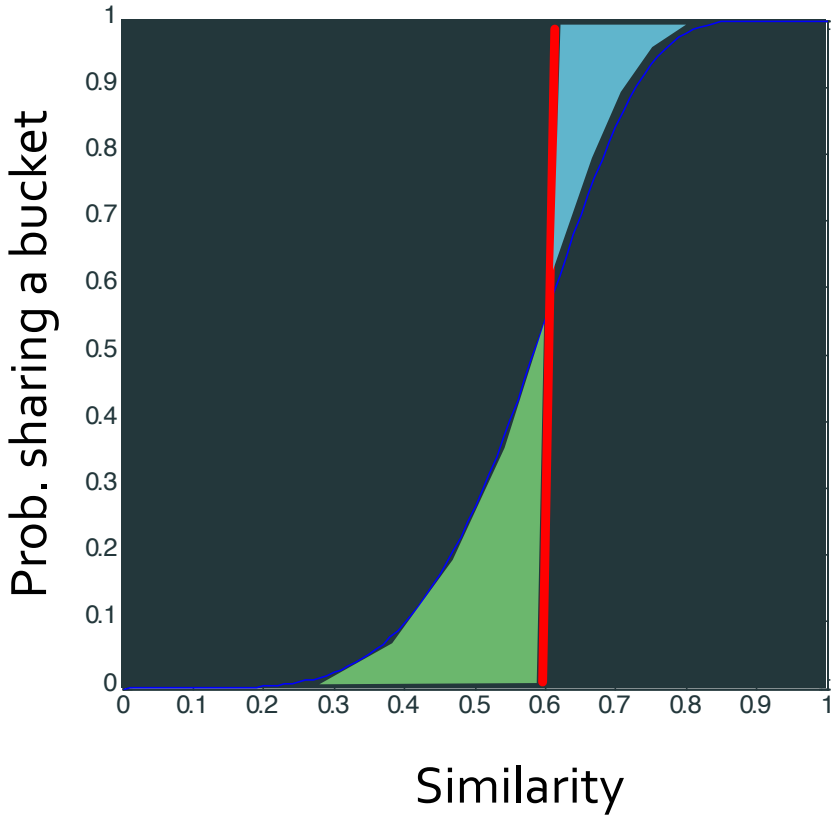
Say $D_1$ and $D_2$ are only 30% similar, $b = 20$, $r = 5$

- prob. $D_1$,$D_2$ identical in a given band $0.3^5 = 0.00243$
- prob. are similar in at least one of the bands:
  $1 - (1 - 0.00243)^{20} = 0.0474$

Around 4.74% of documents having similarity of 30% end as candidate pairs – false positives (since they are not similar, but we still have to check them)

# Using the S-curve

Have to select $r$ and $b$ to get the best curve – one which minimizes the **false negatives** (blue) and **false positives** (green)



# Putting it All Together

Outline of the steps for similar items:

1. pick a value of $k$, and construct $k$-shingles for each documents
2. pick a length $n$ for the min-hash signatures (number of permutations)
3. choose a threshold $t$, along with $b$ and $r$ such that $br = n$ and $t = (1/b)^{1/r}$
4. construct candidate pairs by applying LSH
5. check each candidate pairs in main memory for similarity

# To Go Further

Other **similarity/distance functions** with various application (Sec. 3.5 of [Leskovec et al., 2020])

The mathematical theory behind LSH functions and applying LSH to other similarities (Sec. 3.6 and 3.7 of [Leskovec et al., 2020], [Indyk et al., 1997])

# Acknowledgments

The contents and some figures taken from Chapter 3 of [Leskovec et al., 2020]. `https://www.mmds.org/`

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